Finite constraint set receding horizon quadratic control

Daniel E. Quevedo*†, Graham C. Goodwin and José A. De Doná

School of Electrical Engineering and Computer Science, The University of Newcastle, Callaghan, NSW 2308, Australia

SUMMARY
This contribution addresses the problem of discrete time receding horizon quadratic control for plants whose input is restricted to belong to a finite set. We also study the dynamics of the resulting closed-loop system. Based upon the geometry of the underlying quadratic programme, a finitely parametrized expression for the control law is derived, which makes use of vector quantizers. Alternatively, the control law can be formulated by means of a polyhedral partition of the state space, which is closely connected with the partition induced when considering saturation-like constraints. Exact analytic expressions for the partition can be developed, therefore avoiding the need for on-line optimization. The closed-loop system, comprising controller and plant, exhibits highly nonlinear dynamics, due to the finite set restriction. Asymptotic stability only holds for very special cases. In general, this notion is too strong. Nevertheless, ultimate boundedness of state trajectories is often achieved. Tools for determining positively invariant sets, hence ensuring ultimate boundedness, are presented. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: predictive control; receding horizon; constrained control; finite sets; ultimate boundedness

1. INTRODUCTION

Control systems in which the plant input is restricted to belong to a set containing only a finite number of elements are widespread. The most extensively studied case corresponds to on-off control and relay-feedback, where the constraint set contains only two elements, see e.g. References [1, 2]. However, this situation is also implicit in digital control systems which are affected by quantization, see e.g. References [3–9]. It is also related to control with communication constraints, as studied e.g. in References [10–13]. In these cases, the alphabet (borrowing this terminology from the communication literature) may contain a larger number of elements.

In addition, finite alphabet control laws form a precursor to hybrid systems as documented e.g. in References [14–17]. Indeed, the problem under study here can be cast in that framework. Nevertheless, its peculiar nature allows us to develop a specific methodology which embellishes a distinctive approach.
On the other hand, model predictive control schemes constitute a useful tool for controlling constrained systems [18,19]. Significant research has been carried out for the case of linear plants with $\ell_\infty$ (saturation-like) constraints, see e.g. References [20–22]. However, with the exception of an early application study related to on-off control [23], the finite-alphabet case has only recently been documented [24,25].

This contribution studies the discrete time receding horizon quadratic control problem with a finite constraint set on the input. It presents, and extends, the results included in our conference papers [24,25]. A closed form expression for the control law is derived by exploring the geometry of the underlying quadratic programme. The solution can also be characterized by means of a partition of the state–space, which is closely related to the partition induced by the $\ell_\infty$ constrained solution, as developed in Reference [20]. As a consequence, the controller can be implemented without relying upon on-line optimization. Furthermore, the insight obtained from this viewpoint into the nature of the control law can be used to study the dynamic behaviour of the closed-loop system. Here, in general, the notion of asymptotic stability is too strong and is, therefore, complemented by that of ultimate boundedness.

An outline of the remainder of the paper is as follows: The problem is formulated in the next section. Its solution is developed in Section 3. Section 4 provides details of the corresponding state–space partition. In particular, we explore the connection between the partitions induced by the finite-alphabet- and the $\ell_\infty$-constrained solutions. The dynamics of the resulting controlled system is studied in Section 5. This leads to the need to determine positively invariant sets (Section 6). A case study is included in Section 7. Section 8 draws conclusions.

2. GENERAL PROBLEM FORMULATION

Consider a plant with scalar input $u(k)$ and state vector $x(k) \in \mathbb{R}^n$ described by

$$x(k + 1) = Ax(k) + Bu(k)$$

(1)

The input is restricted to belong to the finite set

$$\mathcal{U} = \{s_1, s_2, \ldots, s_{n_U}\}$$

(2)

where $s_i < s_{i+1} \in \mathbb{R}$, $i = 1, 2, \ldots, n_U - 1$.

In this contribution, we concentrate upon the receding horizon quadratic regulator problem with finite set constraints. Thus, given the state $x(k) = x$, we seek the optimizing sequence of present and future control inputs contained in

$$u^*(x) = \arg \min_{u(k) \in \mathcal{U}^N} V_N(x, u(k))$$

(3)
where

\[
\mathbf{u}(k) = \begin{bmatrix}
    u(k) \\
    u(k + 1) \\
    \vdots \\
    u(k + N - 1)
\end{bmatrix}, \quad \mathbb{U}^N = \mathbb{U} \times \cdots \times \mathbb{U}
\]  

(4)

In (3), \( V_N \) is the finite horizon quadratic cost functional:\nopagebreak

\[
V_N(x, \mathbf{u}(k)) = \sum_{k=0}^{k+N-1} \{\|x(t)\|_Q^2 + \|u(t)\|_R^2\}
\]  

(5)

with \( Q = Q^T > 0, P = P^T > 0, R = R^T > 0 \) and where \( x(k) = x \).

Note that, in the formulation of \( V_N(\cdot, \cdot) \), predictions of future plant states are used. In order to keep the notation simple, we have not introduced a notation, such as e.g. \( x(t|k) \), which allows to distinguish predictions from actual plant states, hoping that this differentiation will be clear from the context.

The minimization of (5) subject to the finite set constraint on \( \mathbf{u}(k) \) and the plant dynamics expressed in (1) yields the optimal sequence \( \mathbf{u}^*(x) \). It is a function only of the current state value \( x(k) = x \).

In the case of the receding horizon regulator under study here, only the first control action, namely

\[
\mathbf{u}^*(x) = [1 \ 0 \ \cdots \ 0] \mathbf{u}^*(x)
\]  

(6)

is applied. At the next time instant, the optimization is repeated with a new initial state and the finite horizon window shifted by one. This procedure constitutes a closed-loop control law, see e.g. Reference [18] for an introduction to receding horizon control.

Remark 1 (Multiple Inputs)

Whilst for ease of exposition we restrict here to the scalar case, the extension to the multiple-input case presents no conceptual difficulties and, indeed, has been recently reported in Reference [26].

In the next section we present a closed-form expression for \( \mathbf{u}^*(x) \). As developed in later sections of this paper, this allows one to characterize the control law as a partition of the state space and provides a tool for studying the dynamic behaviour of the resulting closed-loop system.

\( \|v\|_S^2 \) denotes \( v^T S v \), where \( v \) is any vector and \( S \) is a matrix.
3. NEAREST NEIGHBOUR CHARACTERIZATION OF THE SOLUTION

Since the constraint set $\mathbb{U}^N$ is finite, the optimization problem (3) is non-convex and solving it is certainly not trivial. In order to obtain a solution, it is useful to vectorize the cost function (5) as follows:

Define

$$x(k) \triangleq \begin{bmatrix} x(k+1) \\ x(k+2) \\ \vdots \\ x(k+N) \end{bmatrix}, \quad \Phi \triangleq \begin{bmatrix} B & 0 & \cdots & 0 & 0 \\ AB & B & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & AB & B \end{bmatrix}, \quad \Lambda \triangleq \begin{bmatrix} A \\ A^2 \\ \vdots \end{bmatrix}$$

(7)

so that, given $x(k) = x$ and by iterating Equation (1), the predictor $x(k)$ satisfies

$$x(k) = \Phi u(k) + \Lambda x$$

(8)

Hence, functional (5) can be re-written as

$$V_N(x, u(k)) = \overline{V}_N(x) + (u(k))^T W u(k) + 2(u(k))^T F x$$

(9)

where

$$W \triangleq \Phi^T Q \Phi + R \in \mathbb{R}^{N \times N}, \quad F \triangleq \Phi^T Q A \in \mathbb{R}^{N \times n}$$

and $\overline{V}_N(x)$ does not depend upon $u(k)$. (Since $W > 0$ and the minimization of (9) depends on the vector of parameters $x(k)$, this problem is called a multi-parametric quadratic programme [21].)

Remark 2 (Unconstrained optimum)

By direct calculation, it follows that the minimizer to (9), without taking into account any constraints on $u(k)$, is

$$u_{u}^*(x) = - W^{-1} F x$$

(10)

Although here the main emphasis is on the finite-set constrained case, this well-known expression will be used in the remainder of the paper on several occasions, since some properties of the constrained case can be formulated by using the unconstrained minimizer (10).

Based upon the geometry of (9), we recently presented, in the conference paper [24], a closed-form expression for $u^*(x)$. This result is stated in Theorem 1 below. It requires the following definition of a nearest neighbour vector quantizer. (A thorough treatment of vector quantizers and their features can be found in Reference [27].)
**Definition 1** (Nearest neighbour vector quantizer)

Given a countable (not necessarily finite) set of non-equal vectors \( \mathcal{B} = \{b_1, b_2, \ldots \} \subset \mathbb{R}^n \), the nearest neighbour quantizer is defined as a mapping \( q_\mathcal{B} : \mathbb{R}^n \rightarrow \mathcal{B} \) which assigns to each vector \( c \in \mathbb{R}^n \) the closest element of \( \mathcal{B} \) (as measured by the Euclidean norm), i.e. \( q_\mathcal{B}(c) = b_i \in \mathcal{B} \) if and only if \( c \) belongs to the region\(^6\)

\[
\{ c \in \mathbb{R}^n : ||c - b_i||^2 \leq ||c - b_j||^2, \forall b_j \neq b_i, b_j \in \mathcal{B} \}
\]

\[
\setminus \{ c \in \mathbb{R}^n : \exists j < i \text{ such that } ||c - b_i||^2 = ||c - b_j||^2 \} \quad (11)
\]

Note that in the special case, when \( n_\mathcal{B} = 1 \), the quantizer defined above reduces to the standard scalar quantizer.

**Remark 3** (Simplified definition)

In the foregoing definition, the zero measure set of points which satisfy (11) with equality have been arbitrarily assigned to the element having the smallest index. This is done in order to avoid ambiguity in the case of frontier points, that is, points which are equidistant to two, or more, elements of \( \mathcal{B} \). If this aspect does not matter, then Expression (11) can be simplified to

\[
\{ c \in \mathbb{R}^n : ||c - b_i||^2 \leq ||c - b_j||^2, \forall b_j \neq b_i, b_j \in \mathcal{B} \}
\]

Given Definition 1, we can state the solution to (3). This result constitutes one of the main contributions of this work.

**Theorem 1** (Closed-form solution)

Suppose \( \mathcal{U}^N = \{v_1, v_2, \ldots, v_r\} \), where \( r = n_\mathcal{U}^N \), then the optimizer \( u^*(x) \) in (3) is given by

\[
u^*(x) = W^{-1/2} q_\mathcal{U}^N(-W^{-T/2}F_x)
\]

where the nearest neighbour quantizer \( q_\mathcal{U}^N(\cdot) \) maps \( \mathbb{R}^N \) to \( \hat{\mathcal{U}}^N \), defined as

\[
\hat{\mathcal{U}}^N = \{\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_r\}, \quad \tilde{v}_j = W^{1/2}v_j, \quad v_j \in \mathcal{U}^N
\]

**Proof**

For fixed \( x \), the level sets of the cost (9) are ellipsoids in the input sequence space \( \mathbb{R}^N \). These are centred at the point \( u^*_N(x) \) defined in (10). Thus, the optimization problem (3) can be geometrically interpreted as follows: Find the point \( u(k) \in \mathcal{U}^N \), which belongs to the smallest ellipsoid defined by (9) (i.e. the point which provides the smallest cost whilst satisfying the constraints).

In order to simplify the problem, we introduce the change of variables:

\[
u(k) = W^{1/2}u(k)
\]

which transforms \( \mathcal{U}^N \) into \( \hat{\mathcal{U}}^N \) defined in (14). The optimizer \( u^*(x) \) can be defined in terms of this auxiliary variable as

\[
u^*(x) = W^{-1/2}\arg \min_{\mu(k) \in \hat{\mathcal{U}}^N} J_N(x, \mu(k))
\]

\[^6||v||^2 \text{ denotes } v^Tv, \text{ where } v \text{ is any vector.}\]
where

\[ J_N(x, \mu(k)) = \mu(k)^T \mu(k) + 2(\mu(k))^T W^{-1/2} F x \]  

(17)

The level sets of \( J_N \) are spheres in \( \mathbb{R}^N \), centred at

\[ \mu_{uc}^*(x) = -W^{-1/2} F x \]  

(18)

Hence, the constrained optimizer to (17), is given by the nearest neighbour to \( \mu_{uc}^*(x) \), namely

\[ \arg \min_{\mu(k) \in \mathbb{R}^N} J_N(x, \mu(k)) = q_{\mathbb{U}^N}(-W^{-1/2} F x) \]  

(19)

Result (13) follows by substituting (19) into (16). (It should be emphasized here that transformation (15) can also be used in order to solve the saturation-like constrained receding horizon problem, see Reference [20].)

It is worth noting, that with \( N > 1 \), the optimizer \( u^*(x) \) provided in Theorem 1 is, in general, different to the sequence obtained by direct quantization of the unconstrained minimum (10), i.e. \( q_{\mathbb{U}^N}(\mu_{\text{uc}}^*(x)) \).

As a consequence of Theorem 1, the receding horizon controller (6) satisfies

\[ u^*(x) = [1 \ 0 \ \cdots \ 0] W^{-1/2} q_{\mathbb{U}^N}(-W^{-1/2} F x) \]  

(20)

This solution can be illustrated as the composition of the following transformations:

\[ x \in \mathbb{R}^n \xrightarrow{\ -W^{-1/2} F} \mu_{uc}^*(x) \in \mathbb{R}^N \xrightarrow{\ W^{-1/2} q_{\mathbb{U}^N}} u^*(x) \in \mathbb{U}^N \xrightarrow{\ [10 \cdots 0]} u^*(x) \in \mathbb{U} \]  

(21)

It is worth noting that \( q_{\mathbb{U}^N}(\cdot) \) is a memoryless nonlinearity, so that (20) corresponds to a time-invariant nonlinear state feedback law. In a direct implementation, at each time step, the quantizer needs to perform \( r \) comparisons. However, more efficient search algorithms exist, see e.g. Reference [27, Section 10.4].

4. THE INDUCED PARTITION OF THE STATE SPACE

Expression (11) partitions the domain of the quantizer into polyhedra, called Voronoi regions [27]. Since the constrained optimizer \( u^*(x) \) in (13) (see also (21)) is defined in terms of \( q_{\mathbb{U}^N}(\cdot) \), an equivalent partition of the state–space can be derived, as shown below:

**Theorem 2**

The constrained optimizing sequence \( u^*(x) \) in (13) can be characterized as

\[ u^*(x) = v_i \iff x \in \mathcal{R}_i \]

where

\[ \mathcal{R}_i = \{ z \in \mathbb{R}^n : 2(v_i - v_j)^T F z \leq ||v_j||^2_W - ||v_i||^2_W, \ \forall v_j \neq v_i, v_j \in \mathbb{U}^N \} \]

\[ \{ z \in \mathbb{R}^n : \exists j < i, \text{ such that } 2(v_i - v_j)^T F z = ||v_j||^2_W - ||v_i||^2_W \} \]  

(22)
Proof
From Expressions (13) and (14) it follows that $u^*(x) = v_i$ if and only if $q_{U^V}(-W^{-T/2}Fx) = \tilde{v}_i$. On the other hand,

$$|| - W^{-T/2}Fx - \tilde{v}_i ||^2 = ||W^{-T/2}Fx||^2 + ||\tilde{v}_i||^2 + 2\tilde{v}_i^TW^{-T/2}Fx$$

so that

$$|| - W^{-T/2}Fx - \tilde{v}_i ||^2 \leq || - W^{-T/2}Fx - \tilde{v}_j ||^2$$

holds if and only if

$$2(\tilde{v}_i - \tilde{v}_j)^TW^{-T/2}Fx \leq ||\tilde{v}_j||^2 - ||\tilde{v}_i||^2$$

This inequality together with expressions (14) and (11) shows that

$$q_{U^V}(-W^{-T/2}Fx) = \tilde{v}_i$$

if and only if $x$ belongs to the set $\mathcal{R}_i$ defined in (22). This fact completes the proof. \qed

4.1. Properties of the partition

The $n_U^V$ regions $\mathcal{R}_i$ defined in (22) are polyhedra. Without taking into account constraint borders, they can be written in a compact form as

$$\mathcal{R}_i = \{x \in \mathbb{R}^n: D_i x \leq H_i\}$$

where the rows of $D_i$ are equal to all terms $2(v_i - v_j)^T F$ as required, while the vector $H_i$ contains the scalars $||v_j||^2_W - ||v_i||^2_W$.

Remark 4 (Non-adjacent regions)

Some of the inequalities in (22) may be redundant. In these cases, the corresponding regions do not share a common edge, i.e. are not adjacent. This phenomenon is illustrated in Figure 1, where the regions $\mathcal{R}_2$ and $\mathcal{R}_3$ are not adjacent. The inequality separating them is redundant.

![Figure 1. Partition of the transformed input sequence space with $N = 2$ (solid lines) and two examples of $-W^{-T/2}Fx$, $x \in \mathbb{R}$ (dashed lines).](image)
Remark 5 (Empty regions)
Depending upon the matrix $W^{-T/2}F$, some of the regions $\mathcal{R}_i$ may be empty. This might happen, in particular, if $N > n$. In this case, the rank of $F$ is equal to $n$ and the transformation $W^{-T/2}F$ does not span the entire space $\mathbb{R}^N$. Figure 1 illustrates this for the case $n = 1$, $n_U = 2$ and $N = 2$. As can be seen, depending on the unconstrained optimum locus given by the (dashed) line $-W^{-T/2}Fx$, $x \in \mathbb{R}$, there exist situations in which some sequences $\hat{v}_j$ will never be optimal, yielding empty regions in the state space (see also Figure 4 in Section 7.1, below).

On the other hand, if the pair $(A, B)$ is completely controllable and $A$ is invertible, then the rank of $F$ is equal to $\min(N, n)$.

4.2. The receding horizon case
In the receding horizon case of (20), only $n_U$ instead of (at most) $n_N$ regions are needed to characterize the control law. Each of these regions is given by the union of all regions $\mathcal{R}_i$ corresponding to vertices $v_i$ having the same first element. The appropriate extension of Theorem 2 is presented below. This result follows directly from Theorem 2 and hence, it is stated without a proof.

Corollary 1 (State space partition)
Let the constraint set $U$ be given in (2) and consider the following partition into equivalence classes:

$$U^N = \bigcup_{i=1}^{n_U} U_N^i$$

where

$$U_N^i \triangleq \{v \in U^N: [1 \ldots 0] v = s_i\}$$

Then, the receding horizon control law (20) is equivalent to

$$u^*(x) = s_i \quad \text{if} \quad x \in \mathcal{X}_i, \quad i = 1, 2, \ldots n_U$$

(23)

Here, the polyhedra $\mathcal{X}_i$ are given by

$$\mathcal{X}_i \triangleq \bigcup_{j, y \in U_N^i} \mathcal{X}_{ij}$$

where

$$\mathcal{X}_{ij} \triangleq \{z \in \mathbb{R}^n: 2(v_j - v_k)^T F z \leq \|v_k\|^2_W - \|v_j\|^2_W, \forall v_k \in U_N^i \setminus U_N^j\}$$

$$\setminus \{z \in \mathbb{R}^n: \exists v_k \in U_N^i \setminus U_N^j, k < j, \text{ such that } 2(v_j - v_k)^T F z = \|v_k\|^2_W - \|v_j\|^2_W\}$$

It should be emphasized that this description requires less evaluations of inequalities than the direct calculation of the union of all $\mathcal{R}_j$ (as defined in (22)) with $v_j \in U_N^j$, since inequalities corresponding to internal borders are not evaluated. Moreover, as in Remark 3, the definition of $\mathcal{X}_i$ (and of $\mathcal{R}_i$) can be simplified if the ambiguity problem is not taken care of.

The state space partition obtained can be calculated off-line. It is related to the partition that characterizes the $\ell_\infty$ constrained case, as detailed in the following section.

4.3. Relationship to saturation constraints

If in the set-up described in Section 2, the input is not constrained to belong to a finite set $\mathbb{U}$, but instead, needs to satisfy the $\ell_\infty$ (saturation-like) restriction:

$$-\Delta \leq u(k) \leq \Delta, \quad \forall k$$

(24)

where $\Delta \in \mathbb{R}$ is fixed, then a convex optimization problem is obtained.

For this case, as shown in References [20, 21], the control law can be finitely parameterized in closed loop and calculated off-line. The state space is partitioned into polytopes in which the receding horizon controller is piecewise affine in the state.

The partition of the $\mathbf{u}$-space established by Seron et al. [20] using transformation (15) and a geometric argument similar to the one used in the proof of Theorem 1 is sketched in Figure 2 for the case $N = 2$ and restriction (24). In Figure 2, the polytope $\Theta_o$ is obtained by applying transformation (15) to the region in which the constraints are not active. It is the allowed set. The regions denoted as $\Theta_{vi}$ are adjacent to a hyper-face of $\Theta_o$. The regions $\Theta_{vi}$ share with $\Theta_o$ only a border of lower dimension.

As shown in Section 3, in the finite set constrained case the constrained solution $\mathbf{u}^*$ is related to $\mathbf{u}_{\mathbf{uc}}$ by means of a nearest neighbour quantizer as stated in Equation (13). (For ease of notation, the dependence on $x$ of this and other vectors to follow has not been explicitly included.) A similar result holds in the $\ell_\infty$-constrained case. Given (24), the constrained optimizer, denoted here as $\mathbf{u}_{\mathbf{uc}}^*$, is related to $\mathbf{u}_{\mathbf{uc}}^*$ via a minimum Euclidean distance projection to the allowed set. This result was shown in Reference [20] and can be summarized as follows:

Remark 6 (Projection in the $\ell_\infty$-constrained case)

If $\mathbf{u}_{\mathbf{uc}}^*$ lies inside of $\Theta_o$, then it holds that $\mathbf{u}_{\mathbf{uc}}^* = \mathbf{u}_{\mathbf{uc}}^*$. On the other hand, if $\mathbf{u}_{\mathbf{uc}}^* \notin \Theta_o$, then the constrained solution is obtained by its orthogonal projection onto the border of $\Theta_o$.

Figure 2. Partitions of $\mathbf{u}$-space with $\ell_\infty$, and binary-constraint sets (solid and dashed lines, respectively).
In particular, if the unconstrained solution lies in any of the regions adjacent to a hyper-face of \( \Theta_\alpha \), then \( \mu^* \) is obtained by an orthogonal projection onto the nearest hyper-face (as illustrated in Figure 2 by means of a dotted line).

As a consequence of the foregoing discussion, we obtain the following theorem, which establishes a connection between the partition of the \( \mu \)-space in the \( \ell_\infty \)-constrained case and the Voronoi partition of the quantizer defining the solution with a special finite set constraint.

**Corollary 2** (Relationship between the binary and the \( \ell_\infty \)-constrained case)
Consider the binary constraint set \( \mathbb{U} = \{-\Delta, \Delta\} \) and the region outside of \( \Theta_\alpha \). Then, the borders of the Voronoi partition of the quantizer in (13) are parallel and equidistant to the borders of those regions of the \( \ell_\infty \)-constrained case, which are adjacent to an \( N-1 \)-dimensional hyper-face of \( \Theta_\alpha \). (These regions are denoted in Figure 2 as \( \Theta_{d_i} \).)

**Proof**
From (18), it follows that solution (13) can be stated alternatively as \( u^*(x) = W^{-1/2} q_{\mathbb{U}}(\mu_{\mathbb{U}}^*(x)) \).

The result is a consequence of the fact that the borders of the regions \( \Theta_{d_i} \) are formed by orthogonal projections to \( \hat{v}_i \) and that the Voronoi partition is formed by equidistant hyper-planes which are also orthogonal to the corresponding \( (N-1) \)-dimensional hyper-face of \( \Theta_\alpha \). \( \square \)

This result is illustrated in Figure 2, where the Voronoi partition is depicted via dashed lines. Owing to linearity of the mapping \( W^{-1/2} F \) in (21), the induced partition of the state–space given constraint (24) and the partition defined in (22) are similarly related.

With the above background to the problem in place, the remainder of this paper studies the dynamics obtained when controlling the plant (1) by means of the receding horizon controller (6).

**5. DYNAMICS AND STABILITY ISSUES**

Corollary 1 of Section 4 allows one to describe the closed loop that results when controlling the plant (1) with the law (23) via the following piecewise-affine map:

\[
x(k+1) = g(x(k))
\]

\[
g(x(k)) \triangleq Ax(k) + Bs_i \quad \text{if} \quad x(k) \in \mathcal{X}_i, \quad i = 1, 2, \ldots n_U
\]

Piecewise-affine maps are **mixed mappings** and also form a special class of hybrid systems with underlying discrete-time dynamics, see e.g. References [28, 29] and the references therein. They also appear in connection with some signal processing problems, namely arithmetic overflow of digital filters [30] and \( \Sigma\Delta \)-modulators [31, 32], and have also been studied in a more theoretical mathematical context, see e.g. Reference [33].

Since there exist fundamental differences in the dynamic behaviour of (25), depending on whether the plant (1) is open-loop stable or unstable, i.e. on whether the matrix \( A \) is (strictly) Hurwitz or not, it is convenient to divide the discussion that follows accordingly.
5.1. Stable plants

If plant (1) is stable, then its states are always bounded, when controlled by means of any finite constraint set law. This follows directly from the fact that $U$ is always bounded.

Moreover, it can also be shown by using results from [34], that basically all state trajectories* of (25) either converge towards a fixed point or towards a limit cycle. Both situations can be identified by using Tsypkin’s method [35] described below.

Procedure 1 (Tsypkin’s method)

Consider a candidate plant input sequence of period $p$, $U_p = (s(1), s(2), \ldots, s(p), s(1), \ldots)$ (denoted in what follows as $U_p = (s(1), s(2), \ldots, s(p))$).

The corresponding candidate limit cycle $X_p = (x(1), x(2), \ldots, x(p))$ can be calculated by iterating (25):

$$
\begin{align*}
x(1) &= (I - A^p)^{-1} \sum_{j=1}^{p} A^{p-j} B s(j) \\
x(2) &= Ax(1) + Bs(1) \\
&\vdots \\
x(p) &= Ax(p-1) + Bs(p-1)
\end{align*}
$$

If $X_p$ so obtained is consistent with $U_p$, i.e. if for every $x(i) \in X_p$, it holds that $x(i) \in X_j \Leftrightarrow u(i) = s_j$, then $U_p$ is an admissible input sequence and also $X_p$ is admissible. (Fixed points can be accommodated by setting $p = 1$.)

Which limit cycles are admissible, depends upon the partition of the state space, i.e. on the particular control law.

Note that, since $A$ is assumed to be Hurwitz, it follows that every admissible limit cycle (and fixed point) $X_p$ is surrounded by a non-empty region, called its basin of attraction. States contained in this basin converge towards $X_p$.

The properties stated so far apply to general systems described by (25), where $X_f$ defines any partition of the state space. In contrast, the following theorem is more specific. It utilizes the fact that the control law $u^*(x)$ is optimizing in a receding horizon sense in order to establish a stronger result.

Theorem 3 (Asymptotic stability)

If $A$ is Hurwitz, $0 \in U$ and $P = P^T > 0$ satisfies the Lyapunov equation $A^T PA + Q = P$, then the closed loop (25) is asymptotically stable.

Proof

The proof follows standard techniques used in the model predictive control framework as summarized in Reference [19]. In particular, using the notation of [19], we choose $X_f = \mathbb{R}^n$ and $\kappa_f(x) = 0$, $\forall x \in X_f$. Clearly Assumptions A1–A3 hold and $X_N = \mathbb{R}^n$.

*Exceptions are limited to trajectories that emanate from initial conditions belonging to a zero measure set.
Direct calculation yields
\[ F(f(x, \kappa_f(x))) - \ell(x, \kappa_f(x)) = (Ax + B\kappa_f(x))^T P(Ax + B\kappa_f(x)) - x^T Px + x^T Qx + (\kappa_f(x))^T R\kappa_f(x) \]
\[ = x^T (A^T PA + Q - P) x = 0, \quad \forall x \in X_f \] (27)
so that also A4 is satisfied. Global attractiveness of the origin follows. □

As can be seen, the receding horizon law (6) ensures that the origin is not only a fixed point, but also that it has basin of attraction \( \mathbb{R}^n \).

**Remark 7**
If \( 0 \notin \mathbb{U} \), then the origin is not a fixed point. As a matter of fact, suppose that Assumptions A1–A3 in [19] are satisfied, but that \( 0 \notin \mathbb{U} \), then
\[ F(f(x, \kappa_f(x))) - \ell(x, \kappa_f(x)) = x^T (A^T PA + Q - P) x + 2\kappa_f(x)B^T PAx + (\kappa_f(x))^T (B^T PB + R)\kappa_f(x) \]
Since \( \kappa_f(x) \neq 0 \), this expression is always positive, if \( x \) is close enough to the origin so that A4 is not satisfied.

**Remark 8** (Shifted coordinates)
It is also possible by means of a finite input set constrained control law (1), (2) to steer the plant state asymptotically to any point \( x_i^* \), such that there exists \( x_i \in \mathbb{U} \) which allows one to write \( x_i^* = (I - A)^{-1} Bx_i \).

In order to accomplish this, the cost function (5) needs to be modified by considering *shifted coordinates* as is common when dealing with non-zero constant references in standard model predictive control schemes, see e.g. Reference [36, Section 23.5]. With this modification, the results of this paper, and in particular Theorem 3, apply to the new shifted coordinates.

### 5.2. Unstable plants

In case of strictly unstable plants (1), the situation becomes more involved. Although fixed points and periodic sequences may be admissible, they are basically non-attractive (see Footnote *).

Moreover, with control signals which are limited in magnitude, as is the case with finite-set constraints (and also with saturation-like constraints), there always exists an unbounded region, such that initial states contained in it lead to unbounded state trajectories. This does not mean that every state trajectory of (25) is unbounded. Despite the fact that the unstable open-loop dynamics (as expressed in \( A \)) makes neighbouring trajectories diverge locally, under certain circumstances the control law may keep the state trajectory bounded.

As a consequence of the highly nonlinear (non-Lipschitz) dynamics resulting from the quantizer defining the control law (13), in the bounded case the resulting closed-loop trajectories may be quite complex. In order to analyse them without exploring their fine geometrical structure, it is useful to relax the usual notion of asymptotic stability of the origin. We concentrate on *ultimate boundedness* of state trajectories. This notion refers to convergence towards a bounded region of \( \mathbb{R}^n \), instead of to a point or a specific periodic orbit. (Ultimate
boundedness has also been considered in Reference [37] and by several other authors in the context of practical stability.)

A region, which ensures that once the state enters it, it will not escape in future time, is called positively invariant and can be defined as follows.

**Definition 2** (Positively invariant set)
The set $S_0 \subseteq \mathbb{R}^n$ is said to be positively invariant for system (25), if, for all points $x(t) \in S_0$, the solution $x(k)$ is contained in $S_0, \forall k \geq t$.

More details regarding invariant sets can be found in the survey paper [38]. Finding a positively invariant set is equivalent to solving the ultimate boundedness problem by virtue of Corollary 3 stated below.

**Corollary 3** (Ultimate boundedness)
Suppose (25) has a positively invariant set, $S_0$, then every initial state contained in $S_0$ can be defined as

$$S_0 \triangleq \{x \in \mathbb{R}^n: g^j(x) \in S_0, \forall j \geq i\}, \quad i = 1, 2, \ldots$$

where

$$g^j \triangleq g \circ g \circ \ldots \circ g$$

is the $j$-times iterated map (25), leads in $i$-steps to a bounded trajectory confined in $S_0$.

**Remark 9**
Note that the sets $S_{-i}$ can alternatively be defined via the recursion:

$$S_{-i} = \{x \in \mathbb{R}^n: g(x) \in S_{-(i-1)}\}, \quad i = 1, 2, \ldots$$

This characterization will be used in the Example given in Section 7.2.

**Remark 10**
It follows directly from (28) that $S_{-i} \subseteq S_{-(i+1)}$. In the case of unstable plants, the basin of attraction of $S_0$:

$$S_{-\infty} \triangleq \lim_{i \to \infty} S_{-i}$$

is always bounded.

**Remark 11** (Terminal constraint set)
Positively invariant sets can also be used directly in the formulation of receding horizon control schemes, see References [19, 39]. In them, a cost function similar to (5) is minimized, subject to the additional terminal constraint that the predicted state $x(k + N)$ belongs to a positively invariant set of a local control law.

With finite set constraints (2), ultimate boundedness can be proved for all states steerable to this invariant set. The ultimate boundedness problem is shifted towards a feasibility problem, see e.g. the dissertation [40]. Note that, in order to use this approach, first a positively invariant set needs to be determined. As a consequence of Corollary 3, the argument becomes circular.
Moreover, in this case the closed-form solutions developed in Sections 3 and 4 apply only to the trivial case when the terminal set constraint is redundant.

Having established the relationship between ultimate boundedness and existence of positively invariant sets by means of Corollary 3, in what follows we develop techniques for finding invariant sets. These results were presented earlier by us in the conference paper [25].

6. TECHNIQUES FOR DETERMINATION OF INVARIANT SETS

In this section, two methods for determining positively invariant sets of the map (25) are presented. They utilize the explicit characterizations of $u^*(x)$ obtained in Sections 3 and 4.

6.1. Perturbation technique

Theorem 1 allows one to rewrite the closed-loop mapping (25) as

$$x(k + 1) = \Psi x(k) + B e_u(k)$$

where

$$e_u(k) = u^*(x(k)) - u^*_u(x(k))$$

$$u^*_u(x(k)) = [1 \ 0 \ \cdots \ 0] u^*_u(x(k))$$

$$\Psi = A - B[1 \ 0 \ \cdots \ 0] W^{-1} F$$

and $u^*_u$ is defined in (10). Hence, the finite set constrained loop (25) can be regarded as a perturbation to an unconstrained closed-loop system, which corresponds to the minimization of the cost (5) without constraints on the plant input and is characterized by means of the matrix $\Psi$. As is well known in the standard unconstrained model predictive control context, $\Psi$ can be designed to be stable by a proper choice of $N$ and weighting matrices $P$ and $Q$.

Expression (31) allows one to obtain an analytic characterization of positively invariant sets for (25) by means of Theorem 4 (stated below). It is based upon an idea presented in Reference [41] for $\Sigma\Delta$-modulators and also resembles work related to quantized control systems, see e.g. Reference [5].

Theorem 4 (Analytic positively invariant sets)

Define

$$\bar{U} = \max_{x \in \mathbb{R}} s, \ \underline{U} = \min_{x \in \mathbb{R}} s, \ \bar{\Delta} = \max_{i=1,2,\ldots,n} |s_{i+1} - s_i|$$

$$h = \sum_{i=0}^{\infty} \| T \Psi^i B \|, \ \ T = \left[ \begin{array}{c} 1 \ 0 \ \cdots \ 0 \end{array} \right] W^{-1} F \eta$$

where $\eta \in \mathbb{R}^{n \times n}$ is any vector and $\| \cdot \|$ denotes either the Euclidean- or the infinity-vector-norm, $\|v\| = \max \{|v_i|\}$. Furthermore, consider $N = 1$, suppose that the matrix $\Psi$ is Hurwitz and that $\bar{U} > 0, \ \underline{U} < 0$. 

If for every \( i = 0, 1, \ldots \) it holds that \( |e_u(k - i)| \leq L \), where \( L \geq \Delta_U / 2 \) is fixed, then
\[
|h| \leq 1 + \frac{\min \{\tilde{U}, |\mathbb{U}|\}}{L}
\]  
(33)
is a sufficient condition for the set
\[
\mathcal{S} = \{x \in \mathbb{R}^n : \|Tx\| \leq hL\}
\]  
(34)to be a positively invariant set for the closed loop (25).

**Proof**
For \( N = 1 \), result (13) provides \( u^*(x) = q_U(u^*_w(x)) \). Thus,
\[
|e_u(k + 1)| = |q_U(u^*_w(x(k + 1))) - u^*_w(x(k + 1))| = \min_{s \in \mathbb{U}} |s - u^*_w(x(k + 1))|
\]  
(35)In what follows, we will show that this quantization error is bounded, if the conditions of the theorem are satisfied.

Since \( x(k + 1) = \sum_{i=0}^{\infty} x_i \), the size of \( x(k + 1) \) can be upper bounded by using the triangular inequality and the fact that \( e_u \in \mathbb{R} \) as follows:
\[
\|Tx(k + 1)\| \leq \sum_{i=0}^{\infty} \|T\mathbf{u}^i B_k(k - i)\| \leq \max_{i=0,1,\ldots} \{ |e_u(k - i)| \} \sum_{i=0}^{\infty} \|T\mathbf{u}^i B\| \leq Lh
\]  
(36)Furthermore, the definition of \( T \) and Equation (10) provide the lower bound:
\[
\|Tx(k + 1)\| \geq \|1 \ 0 \ \cdots \ 0 W^{-1} Fx(k + 1)\| = |u^*_w(x(k + 1))|
\]  
(37)so that, by combining with inequalities (36) and (33), the unconstrained minimizer is bounded as
\[
|u^*_w(x(k + 1))| \leq L + \min_{s \in \mathbb{U}} \{\tilde{U}, |\mathbb{U}|\}
\]  
(38)Since, as specified in the preamble to the theorem, the value of \( L \) in this expression satisfies \( L \geq \Delta_U / 2 \), it follows that the quantization error satisfies the bound:
\[
\min_{s \in \mathbb{U}} |s - u^*_w(x(k + 1))| \leq L
\]Owing to (35), also \( |e_u(k + 1)| \leq L \). By induction, it follows that for every \( t \in \mathbb{N} \), \( |e_u(t)| \leq L \). As a consequence of (36), also \( \|T x(k + j)\| \leq L h, \ \forall j \geq 1 \). This fact completes the proof.

**Theorem 4** provides a sufficient condition for existence of positively invariant sets. These regions are ellipsoids, if in (34), \( \| \cdot \| \) is chosen as the Euclidean norm, and polytopes if \( \| \cdot \| \) is the infinity norm. The matrix \( T \) determines their shape.

**Lemma 1**
If \( h \) (as in Theorem 4) or the norm of the steady-state response of the system formed by the triplet \( (\mathbf{u}, B, T) \), i.e. \( \|T(I - \mathbf{u})^{-1} B\| \), is larger than \( 1 + 2 \min \{\tilde{U}, |\mathbb{U}|\} / \Delta_U \), then Theorem 4 is not informative.

**Proof**
Since \( L \geq \Delta_U / 2 \), the right-hand side of condition (33) is upper bounded by the term \( 1 + \min \{\tilde{U}, |\mathbb{U}|\} / L \leq 1 + 2 \min \{\tilde{U}, |\mathbb{U}|\} / \Delta_U \).
On the other hand,
\[
h = \sum_{i=0}^{\infty} \| T \Psi^i B \| = \left\| \sum_{i=0}^{\infty} T \Psi^i B \right\| = \| T(I - \Psi)^{-1} B \|
\]
In either of both cases, (33) cannot be satisfied.

Checking explicitly that \( |e_u(k - i)| \leq L \) for every \( i = 0, 1, \ldots \) as required in Theorem 4, is not computational tractable. However, if the associated linear law is dead-beat, this task can be simplified as stated below:

**Corollary 4**

Consider the same conditions as in Theorem 4. If the linear control law (31) (as expressed in \( \Psi \)) is dead-beat in \( m \) steps, then it is sufficient to check \( |e(k - i)| \leq L \) for \( i = 0, 1, \ldots m - 1 \).

**Proof**

The proof follows the same lines as that of Theorem 4. We simply note that, if the law is dead-beat in \( m \) steps, then \( \Psi^m = 0 \) (\( \Psi \) is nilpotent) so that
\[
\sum_{i=0}^{\infty} \Psi^i B e_u(k - i) = \sum_{i=0}^{m-1} \Psi^i B e_u(k - i)
\]

Unfortunately, the extension of Theorem 4 to the case \( N > 1 \) is not straightforward. Boundedness of \( \|Tx(k + 1)\| \) or of \( |u^{fe}_+(x(k + 1))| \) does not easily ensure that \( |e_u(k + 1)| \) is bounded as well.

Before turning to an algorithmic technique for the determination of invariant sets which can handle any prediction horizon (see the next subsection), we first present a special case where Theorem 4 leads to a definitive and tight conclusion.

**Theorem 5** (First-order case)

Consider the same definitions as in Theorem 4, \( N = 1 \) and \( \bar{U} = -\underline{U} > 0 \). Let \( A \) in (1) be a scalar. Then, given \( L \geq \Delta_k / 2 \),
\[
|A| \leq 1 + \underline{U} / L \quad (39)
\]
is a necessary and sufficient condition for \( \mathcal{S} = \{x \in \mathbb{R} : |x| \leq |B|L\} \) to be a positively invariant set for (25).

**Proof**

Direct calculation yields \( \Psi = 0 \), \( W = 1 \), \( F = A / B = T \) and \( h = |A| \) (In the scalar case both the Euclidean- and the infinity-vector-norm reduce to the absolute value function) so that expressions (36) and (37) yield
\[
|Ax(k + 1) / B| = |Ae_u(k)| = |u^{fe}_+(x(k + 1))| \quad (40)
\]
Hence, if \( |x(k)| \leq |B|L \), then \( |u^{fe}_+(x(k))| \leq |A|L \leq L + \bar{U} \), by condition (39). As a consequence, \( |e_u(k)| \leq L \) so that also \( |x(k + 1)| \leq |B|L \), which proves sufficiency.

Suppose now that \( |A| = \alpha(1 + \bar{U} / L) \) with \( \alpha > 1 \) and that \( |x(k)| = \delta|B|L \leq |B|L \) with \( \delta > 1 / \alpha \). Then, due to (40), \( |u^{fe}_+(x(k))| = |A|\delta L = \alpha\delta(L + \bar{U}) \) so that \( |e_u(k)| = \alpha\delta L + \alpha\delta\bar{U} - \bar{U} \) and
\[ |x(k + 1)| = |B(x \delta L + (x \delta - 1) \tilde{U})| > |B|L. \] As a consequence, \( S \) is not positively invariant. This proves necessity. \( \square \)

For example, consider the first-order case and \( U = \{-1, 0, 1\} \), in which case (39) reduces to \(|A| \leq 1 + 1/L\) with \( L \geq 1/2\). Hence, positively invariant sets exist only for plants with \(|A| \leq 3\). With \( U = \{-1, 1\} \), the condition becomes \(|A| \leq 1 + 1/L\), \( L \geq 1\), ensuring ultimate boundedness only if \(|A| \leq 2\). This particular case has been studied elsewhere as well, see e.g. Reference [42].

Another alternative technique for determination of positively invariant sets is presented below. It utilizes the characterization of the receding horizon control law as a state-space partition as given in Corollary 1.

6.2. Evaluation via the inclusion principle

Definition 2 states that, in discrete time, a set is positively invariant if and only if its image under the map is a subset of itself. Thus, in principle, invariance of a given region can be proved by performing an inclusion test for every point in it. Fortunately, due to the nature of the map considered here, this task can be simplified enormously. For affine mappings such as (25) it holds that, the image of a polytope is a polytope and the image of its interior maps to the interior of the polytope's image. As a consequence, the inclusion test needs to be applied only to a finite number of points. These points are the vertices of the polytope and the intersection of the polytope edges with the switching surfaces. This is illustrated in Figure 3 for \( n_U = 2\). As can be appreciated in this figure, the image of the polytope \( ABCDE \) can be determined by evaluating the mapping \( g(\cdot) \) solely at the vertices \( A, B, C, D, E \) and the intersections \( P, Q, R \). Note that, in loose terms, the points \( P, Q \) and \( R \) have two images each, due to the discontinuity of \( g(\cdot) \).

Based upon this fact, in Reference [43] (see Reference [44] for a related procedure) a numerical algorithm for identifying convex polytopal invariant sets for \( \Sigma \Delta \)-modulators is proposed. We extended this result to system (25) in the conference contribution [25]. The algorithm can be

![Figure 3. Inclusion test of a polytope. The dashed polytopes are the images.](image-url)
summarized as follows:

**Procedure 2** (Algorithm for determining invariant sets)

1. Obtain an experimental limit set by iterating the map until a steady state is reached. Discard the transient and calculate the convex hull of the remaining set. This is the initial candidate polytope.
2. Calculate the image of the candidate. As explained above, due to the piecewise affine nature of the map, only a finite number of points, namely the vertices of the candidate polytope and the intersection of the polytope edges with the switching surfaces, needs to be taken into account. The image obtained consists of several (possibly overlapping) polytopes, see Figure 3.
3. If the image obtained in the previous step is a subset of the candidate polytope, then the latter is a positively invariant set and the algorithm finishes. Otherwise, take the convex hull of the candidate polytope and its image and go to the next step.
4. Use the polytope calculated in the previous step as a new candidate and go to Step 2.

**Remark 12**
The actual implementation contains more details in order to prevent the appearance of unnecessarily fine structure and to accelerate convergence. These embellishments are further described in Reference [43].

Note that sets obtained by the algorithm are, by construction, positively invariant and are not an approximation. Also, since the algorithm will enlarge the candidate set in each iteration, it is useful to keep the experimental limit set in Step 1 reasonably small. To that extent, one should carefully inspect the data in order to decide when transient phenomena have perished.

It should be emphasized here, that if the algorithm does not converge, this does not mean that positively invariant sets do not exist. The search performed is restricted to convex polytopes and, depending on the particular case under study, there may exist positively invariant sets having a different geometry.

7. **EXAMPLE**

As an example, consider system (1) with

\[
A = \begin{bmatrix} 0 & 1 \\ -\rho^2 & 2\rho \cos \theta \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]  

(41)

which has poles at $\rho e^{\pm j\theta}$. Choose $\theta = 1.5$, $P$ and $Q$ as identity matrices and $R = 0$.

7.1. **Stable plant**

Figure 4 includes the partitions of the state space obtained for the stable plant (41) with $\rho = 0.9$ for optimization horizons $N = 2, 3, 4, 5$ as described in Section 4. Here, the constraint set is $\mathbb{U} = \{-1, 1\}$. Note that, for $N = 5$, the regions \( R_2, R_{10}, R_{12}, R_{21}, R_{23} \) and $R_{31}$ are empty.
The receding horizon control law is

$$u^*(x) = \begin{cases} -1 & \text{if } x \in X_1 \\ 1 & \text{if } x \in X_2 \end{cases}$$

where

$$X_1 = \bigcup_{i=2^{N-1}+1,2^{N-1}+2,\ldots,2^N} R_i, \quad X_2 = \bigcup_{i=1,2,\ldots,2^{N-1}} R_i$$

Turning now to the closed-loop dynamics, $$x(k+1) = Ax(k) + Bu^*(x(k))$$, Tsyakin’s method, see Procedure 1, allows one to identify both fixed points corresponding to $$u = -1$$ and 1 and the alternating sequence, $$U_2 = (-1, 1)$$. No other periodic sequences of period up to $$p = 14$$ are admissible.

7.2. Unstable plant

Consider next the case $$\rho = 1.2, N = 4$$ but keep the other parameters as in the previous example. With this choice, the matrix $$A$$ in (41) is not Hurwitz and, although many admissible periodic sequences co-exist, none of them is locally attractive.

Figure 5 illustrates a positively invariant set obtained by means of the algorithm described in Section 6.2. In this case, the algorithm converges after three iterations to this reasonably tight set. Note, that the complexity of this polytope is characterized by its 12 vertices.
Figure 5. Switching curve and positively invariant set determined by the algorithm of Section 6.2 for plant (41) with $\rho = 1.2$. The dots are the initial candidate set. The vertices are marked with stars.

Figure 6. Switching curve, positively invariant set and $S_{\rho_i}$, $i = 1, 2, \ldots, 8$. 
Although computational issues are not the focus of the current paper, we give a rough CPU account for this algorithm. The iterations of steps 2–4 require approximately 1 665 000 floating point operations, which on a Pentium 3 PC running at 500 MHz clock speed, utilize 1.15 s of CPU time.

In order to investigate the basin of attraction of the positively invariant set so obtained, it is useful to calculate its pre-images (see Remark 10). This can be obtained recursively by means of Expression (29). Figure 6 illustrates the result. It contains the regions $S_{-i}$ for $i = 1, 2, \ldots, 8$. Note that all boundaries are constructed from straight line segments, due to the piecewise-linear nature of the map.

The procedure used allows one to identify positively invariant sets for the set-up considered here with $\rho \leq 1.349$. (Simulation studies indicate that ultimate boundedness holds $\forall \rho \leq 1.387$.) In the critical case, $\rho = 1.349$, the algorithm converges after 13 iterations and gives rise to the positively invariant set depicted in Figure 7, which also illustrates its pre-images. In this case, approximately 5 494 000 floating point operations, corresponding to 2.03 s of CPU time, were performed.

Figures 6 and 7 suggest that the explanation of the path to instability in this case is a boundary crisis. As $\rho$ grows, the boundary of the positively invariant set tends to collide with (an approximation of) its basin of attraction. This phenomenon leads to instability.

8. CONCLUSION

In this paper, we have studied the receding horizon quadratic control problem for linear plants whose inputs are restricted to belong to a finite alphabet. In particular, we have provided
closed-form solutions for the control law which can be calculated off-line. Furthermore, we have studied the resulting closed-loop dynamics and provided results which allow one to characterize different types of stability, including the notions of asymptotic stability and ultimate boundedness.

In ongoing work we have utilized the general methodology outlined in the current paper in order to address a number of application areas, including: networked control systems [26], audio quantization [45, 46], digital channel equalization [47] and design of filters with quantized coefficients [48].

REFERENCES